2. Second-Order Linear Ordinary Differential Equations

2.1 Standard Form and Terminology

- **General Form:** $a_0(x)y'' + a_1(x)y' + a_2(x)y = a_3(x)$
- **Standard Form:** $y'' + P(x)y' + Q(x)y = R(x)$
- **Homogeneous and Non-homogeneous:** The ODE is homogeneous if $R(x) \equiv 0$; it is non-homogeneous if $R(x) \not\equiv 0$.

2.2 Superposition (Linearity) Principle Theorem

Theorem: For a homogeneous linear second-order ODE, any linear combination of two solutions is again a solution of the equation on the considered interval.

> **Linear Combination:** $y = c_1 y_1(x) + c_2 y_2(x)$

Warning: This principle holds only for homogeneous linear equations. It does not apply to non-homogeneous linear equations or non-linear equations.

2.3 Linear Independence, Wronskian, and Abel's Formula

- **Linear Independence:** Two functions $y_1(x)$ and $y_2(x)$ are linearly independent on an interval I if $c_1 y_1(x) + c_2 y_2(x) = 0$ for all $x \in I$ implies $c_1 = c_2 = 0$.
- **Wronskian (Wronski Determinant):** For differentiable functions y_1, y_2, the Wronskian is defined as:

> $W(y_1, y_2)(x) = y_1 y_2' - y_2 y_1'$

- **Wronskian Test Theorem:** Two solutions y_1, y_2 of $y'' + P(x)y' + Q(x)y = 0$ on an open interval I are linearly independent iff $W(y_1, y_2)(x) \neq 0$ for all $x \in I$. If they are dependent, $W(y_1, y_2)(x) \equiv 0$ on I.
- **Abel's Formula (Identity) Theorem:** The Wronskian of the homogeneous linear ODE satisfies:

> **Differential Equation:** $W' + P(x)\,W = 0$
> **Integral Form:** $W(x) = W(x_0) * \exp(-\int_{x_0}^{x} P(t)\,dt)$

2.4 Existence and Uniqueness Theorem

IVP Theorem: If P(x), Q(x), and R(x) are continuous functions on an open interval I containing x_0, then the initial value problem:

> **ODE:** $y'' + P(x)y' + Q(x)y = R(x)$
> **Initial Conditions:** $y(x_0) = K_0,\ \ y'(x_0) = K_1$

possesses a unique solution y(x) valid on the entire interval I.

2.5 Fundamental Sets, General and Particular Solutions

- **Fundamental Set:** A pair of linearly independent solutions $\{y_1, y_2\}$ of the homogeneous ODE on I.
- **General Solution (Homogeneous):** $y_c(x) = c_1 y_1(x) + c_2 y_2(x)$ with arbitrary constants.
- **Structure Theorem for Non-homogeneous ODEs:** The general solution of $y'' + P(x)y' + Q(x)y = R(x)$ is written as:

> **Formula:** $y(x) = y_c(x) + y_p(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
> **Where:** $y_p(x)$ is any particular solution of the non-homogeneous equation.

2.6 Reduction of Order Method

Method Formula: If one solution $y_1(x)$ of the homogeneous ODE is known, a second linearly independent solution $y_2(x)$ can be constructed by substituting $y_2(x) = u(x) y_1(x)$.

> **Multiplier:** $u(x) = \int [e^{(-\int P(x)\,dx)} / y_1^2(x)]\,dx$
> **Second Solution:** $y_2(x) = y_1(x) * \int [e^{(-\int P(x)\,dx)} / y_1^2(x)]\,dx$

2.7 Homogeneous Linear ODEs with Constant Coefficients

Characteristic (Auxiliary) Equation: Consider the constant coefficient homogeneous equation: $y'' + ay' + by = 0$.

> **Characteristic Equation:** $r^2 + ar + b = 0$

- **Case 1:** Distinct Real Roots ($a^2 - 4b > 0$). Roots $r_1 \neq r_2$ are real:

> **Basis:** $y_1 = e^{(r_1 x)}, \quad y_2 = e^{(r_2 x)}$
> **General Solution:** $y = c_1 e^{(r_1 x)} + c_2 e^{(r_2 x)}$

- **Case 2:** Real Double Root ($a^2 - 4b = 0$). Root $r = -a/2$ has multiplicity 2:

> **Basis:** $y_1 = e^{(r x)}, \quad y_2 = x e^{(r x)}$
> **General Solution:** $y = (c_1 + c_2 x) e^{(r x)}$

- **Case 3:** Complex Conjugate Roots ($a^2 - 4b < 0$). Roots $r = \alpha \pm i\beta$, where $\alpha = -a/2$ and $\beta = \sqrt{(b - a^2/4)}$:

> **Basis:** $y_1 = e^{(\alpha x)} \cos(\beta x), \quad y_2 = e^{(\alpha x)} \sin(\beta x)$
> **General Solution:** $y = e^{(\alpha x)} [c_1 \cos(\beta x) + c_2 \sin(\beta x)]$

2.8 Method of Undetermined Coefficients

Method Rules: For the non-homogeneous equation $y'' + ay' + by = R(x)$, the particular solution y_p can be guessed based on the form of the driving force $R(x)$.

- **Basic Rule:** The form of y_p corresponds to the structure of $R(x)$.

- **Modification Rule:** If any term in the trial form y_p duplicates a solution in y_c, multiply the entire trial form by x , where s is the multiplicity of the matching root (s = 1 for a simple root, s = 2 for a double root).
- **Sum Rule:** If R(x) is a sum of several terms, R(x) = R_1(x) + R_2(x), solve for y_p1 and y_p2 separately and sum them: y_p = y_p1 + y_p2.